WS ON ABSTRACT ANALYSIS 2014

HOMOMORPHISMS, STRUCTURAL RAMSEY THEORY & LIMITS

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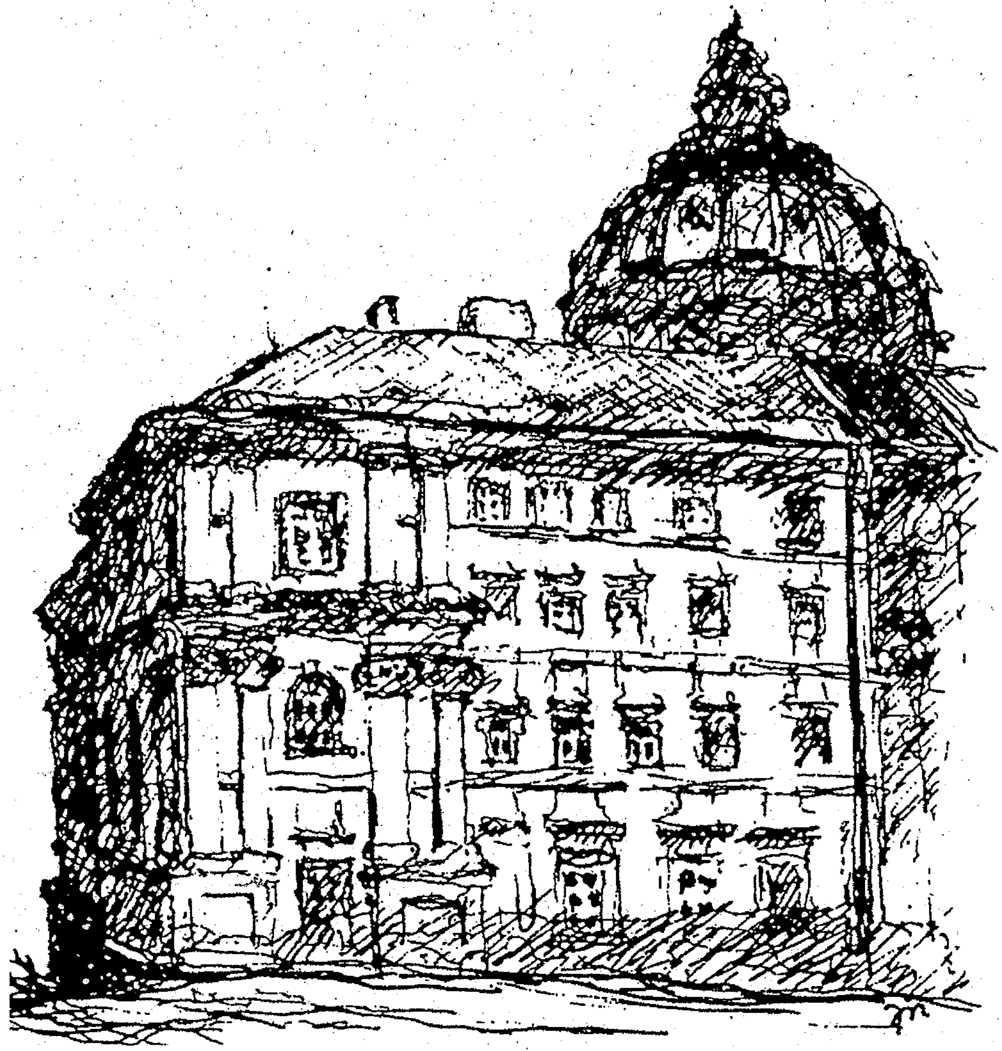
JAN 2014



HOMOMORPHISMS
&
LOGIC
&
LIMITS

(PARTICULARLY FOR
SPARSE STRUCTURES)

ALL STRUCTURES IN A NOWHERE DENSE CLASS

- MAY BE FINITELY APPROXIMATED (WITH ARBITRARY PRECISION)

- SATISFY HOMOMORPHISM PRESERVATION (ROSSMAN; DAWAR; NPOM)

- RELATE TO DESCRIPTIVE COMPLEXITY

HOMOMORPHISM $G \longrightarrow H$ IS
A MAPPING $f : V(G) \longrightarrow V(H)$
SATISFYING :

$$xy \in E(G) \implies f(x)f(y) \in E(H).$$

"EDGE PRESERVING MAPPINGS"

(NOT ONLY GRAPHS ; FINITE RELATIONAL SYSTEMS)

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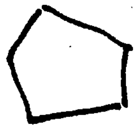
$$f : G \longrightarrow H \quad \equiv \quad H\text{-COLOURING}$$

$$G \longrightarrow K_3 \quad \equiv \quad 3\text{-COLOURING}$$

$G \longrightarrow$  $\equiv C_5$ - COLOURING

$G \longrightarrow$  $\equiv$ PETERSEN - COLOURING

$\vdots$

WHEN IS H-COLOURING HARD PROBLEM

THM (HELL, N. 1990)

H-COLOURING HARD

$\Updownarrow$

H NON-BIPARTITE

OTHER PROOFS

BULATOV
SIGGERS
BARTO-KOZIK
KUN-SZEGEDY

ALL HARD

H-CRITICAL GRAPH G

$\equiv$

$G \not\to H$

$G' \longrightarrow H$

FOR EVERY PROPER SUBGRAPH

WHEN ARE THERE FINITELY MANY H-CRITICAL GRAPHS ?

$\equiv$

DO THERE EXISTS $F_1, \dots, F_t$ SUCH THAT FOR EVERY G

$$\begin{matrix} F_1 \not\to G \\ F_2 \not\to G \\ \vdots \\ F_t \not\to G \end{matrix} \iff G \longrightarrow H$$

$H =$  $= \vec{T}_3$

THE ONLY H-CRITICAL IS

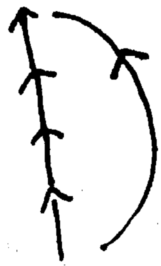
 $\vec{P}_3$

$$\vec{P}_k \not\to G \iff G \to \vec{T}_k$$

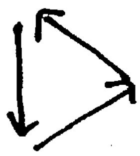
GALLAI, HASSE, ROY, VITAVER

$H =$ 

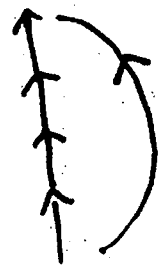
INFINITELY MANY H-CRITICAL

H = 

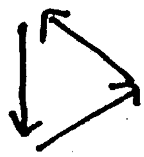
INFINITELY MANY H-CRITICAL

H = 

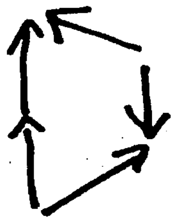
INFINITELY MANY H-CRITICAL

H = 

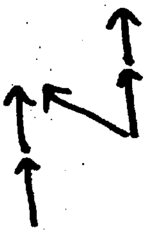
INFINITELY MANY H-CRITICAL

H = 

INFINITELY MANY H-CRITICAL

H = 

THE ONLY H-CRITICAL



KOMÁREK

MORE FORMALISM

$F \not\to \; = \{A : F \not\to A\}$

FORB(F)

FORB$(\mathcal{F})$ FOR $\mathcal{F} = \{F_1, \dots, F_t\}$.

$\boxed{\text{THM}}$ ANY NP PROBLEM POLYNOMIALLY EQUIVALENT TO MEMBERSHIP FORB$(\mathcal{F}')$ FOR A LIFT (EXPANSION τ' OF τ)

(KUN, N.)

$\boxed{\text{THM}}$ ANY HOMOMORPHISM CLOSED FIRST ORDER DEFINABLE CLASS IS OF FORM $\mathcal{F} \longrightarrow$

(ROSSMAN)

$\boxed{\text{FINITE HOM-DUALITY}}$ (N., PULTR)

$$\mathrm{FORB}(\mathcal{F}) = \mathrm{CSP}(D)$$

(EQUATION OF CLASSES, GAME OF ALTERNATIVES:

FOR EVERY G HOLDS:

$F \not\to G$ FOR ALL $F \in \mathcal{F}$

IFF

$G \longrightarrow D$

$\mathcal{F}$, $F \in \mathcal{F}$ ... FORBIDDEN GRAPHS

D ... DUAL

JUST EXPRESSES THE FACT THAT THERE ARE FINITELY MANY D-CRITICAL GRAPHS.

FINITE DUALITIES CHARACTERIZED

- COMBINATORICS KOMÁREK N., TARDIF
  ($\mathcal{F}$ A SET OF TREES)
  (D^2 DISMANTABLE) LAROSE, LOTTEN, TARDIF

- LOGIC ATSERIAS ROSSMAN
  (ONLY FO DEFINABLE CSP)

- HOMOMORPHISM POSET

GAPS CUTS BOUNDS

(HEYTING POSETS)

$G \longrightarrow H$ MAY BE VIEWED AS QUASIORDER

DEFINED ON THE CLASS OF ALL FINITE GRAPHS

IF WE CONSIDER NON-ISOMORPHIC CORE GRAPHS THEN WE GET PARTIAL ORDER

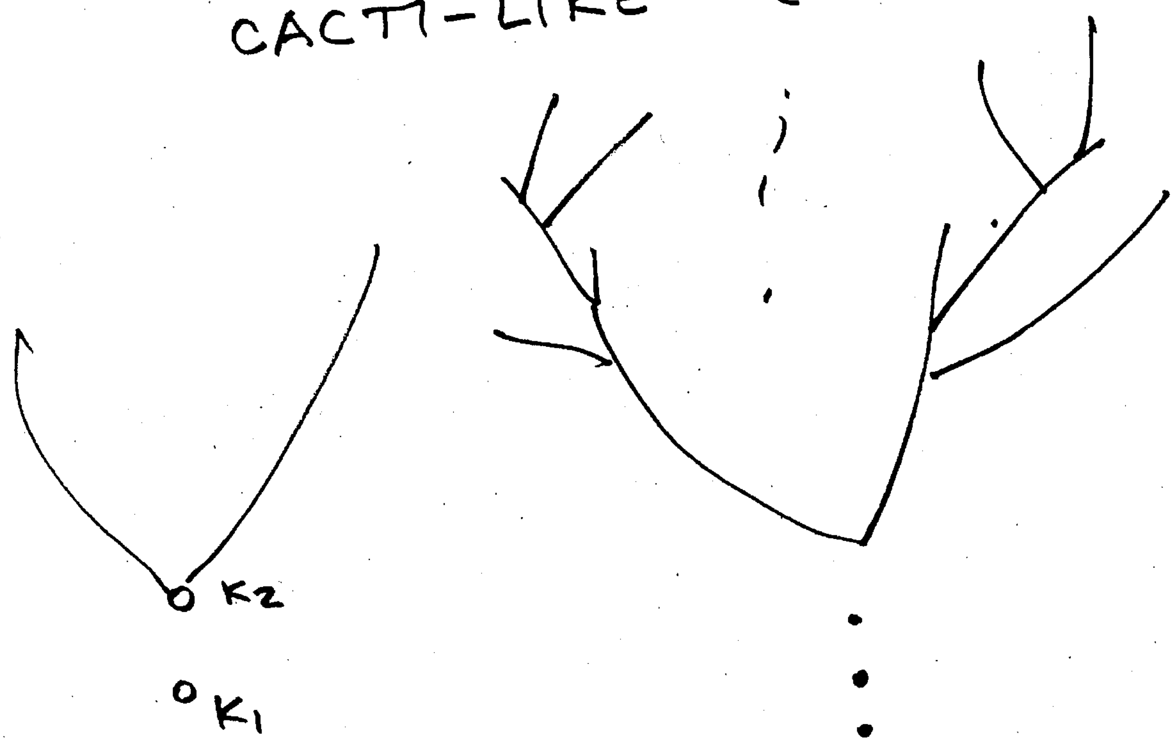
$\mathcal{H}$

HOMOMORPHISM ORDER

SPECTACULAR PROPERTIES OF HOMOMORPHISM ORDER $\mathcal{H}$

- DENSE
- UNIVERSAL
- ∞ CONNECTED (UNDIRECTED)

CACTI-LIKE (DIRECTED)



UNDIRECTED

DIRECTED

RESTRICTED VERSION OF DUALITIES

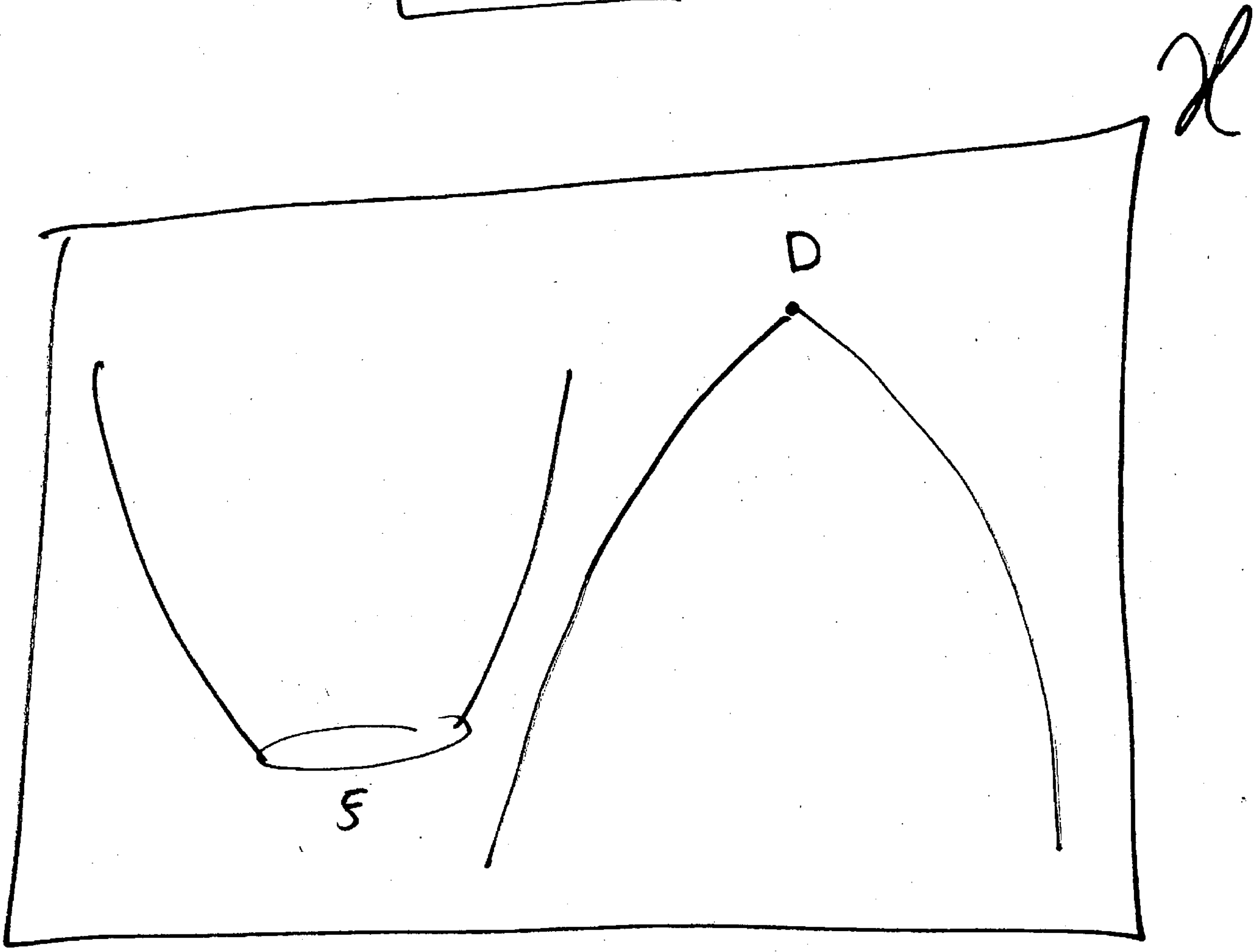
$\mathcal{C}$- RESTRICTED DUALITY

$\mathcal{C}$ A CLASS OF GRAPHS

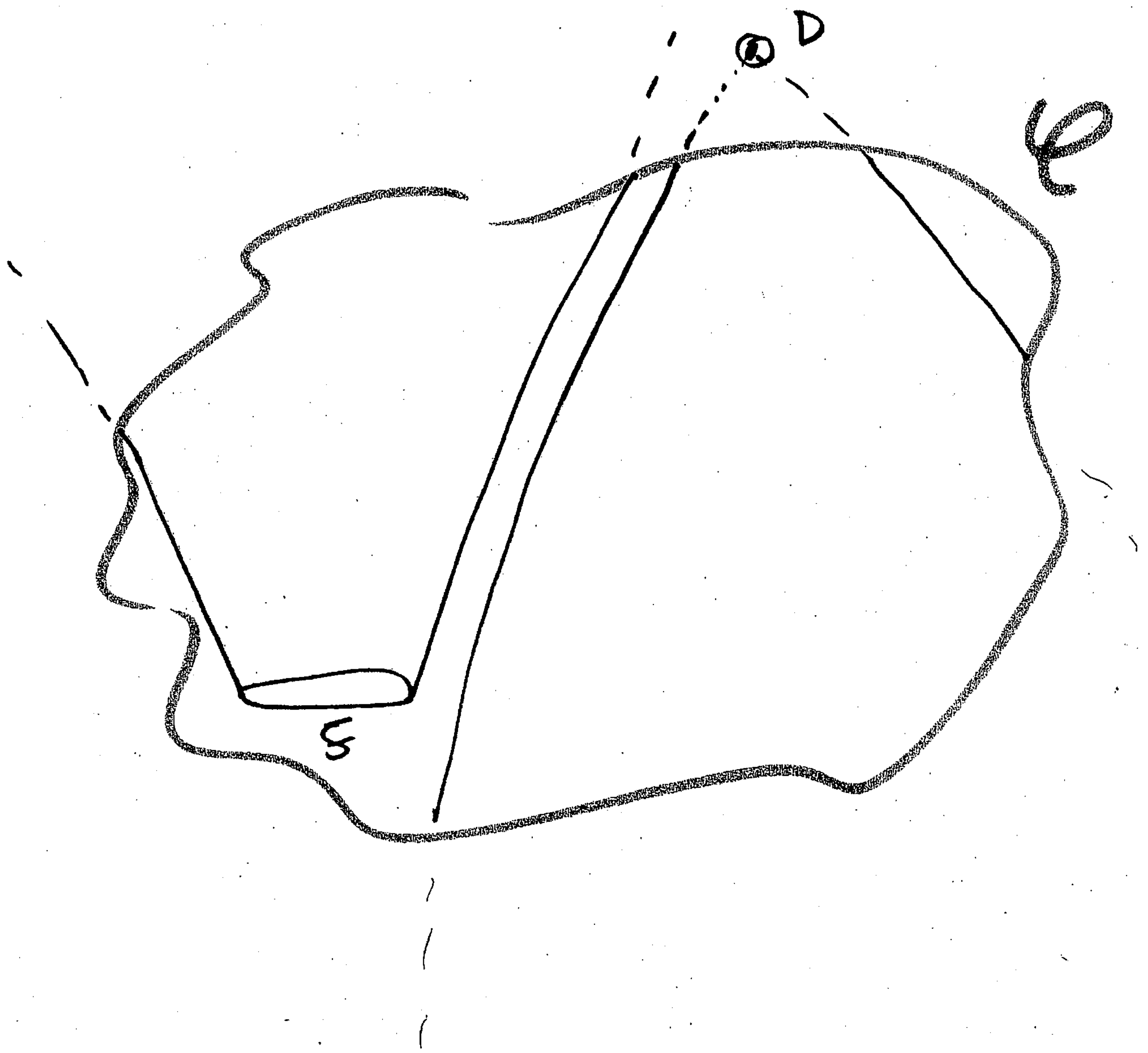
$\mathcal{F} \subseteq \mathcal{C}$

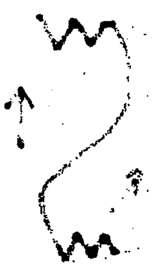
$$\boxed{\mathrm{FORB}(\mathcal{F}) \cap \mathcal{C} = \mathrm{CSP}(D) \cap \mathcal{C}}$$

DUALITY



RESTRICTED DUALITY





$\mathcal{C}$ HAS $\boxed{\text{ALL}}$ RESTRICTED DUALITIES

IFF

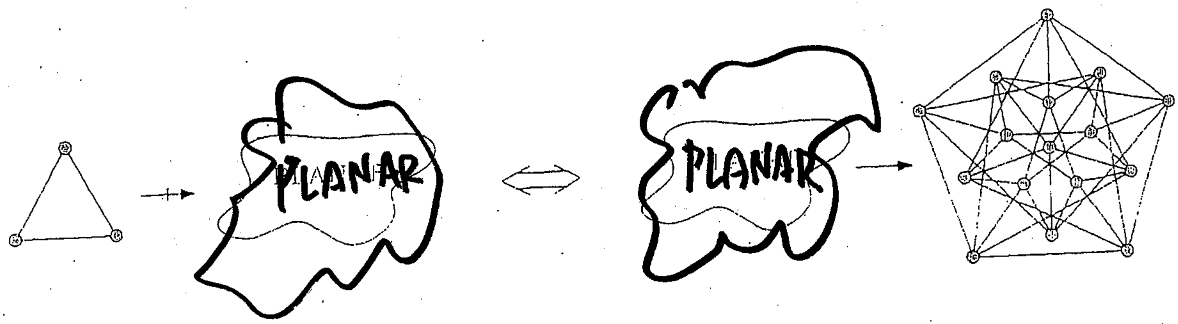
FOR EVERY FINITE SET CONNECTED $\mathcal{F} \subseteq \mathcal{C}$

THERE EXISTS $D_{\mathcal{F}}$ SUCH THAT

$$\mathcal{C} \cap \mathrm{FORB}(\mathcal{F}) = \mathcal{C} \cap \mathrm{CSP}(D_{\mathcal{F}})$$

— BOUNDED DEGREE GRAPHS HAVE ALL RESTRICTED DUALITIES (HAGGKVIST, HELL)

— PLANAR GRAPHS HAVE ARD (N., P. OSSONA DE MENDEZ)



PLANAR – RESTRICTED
DUALITY

WHICH CLASSES HAVE

ARD ?

CHARACTERIZATION BY

— METRIC PROPERTIES OF HOMOMORPHISM ORDER

— ORIENTED AND ACYCLIC LIFTS

— BY SUBDIVISIONS

— BY FO DEFINABILITY

(MODULO WEAK ERDÖS-HAJNAL CONJECTURE)

$$\mathrm{dist}(A,B) = 2^{-k}$$

$$k = \min\left\{ |C| : \begin{array}{c} C \longrightarrow A \ \& \ C \not\longrightarrow B \\ \text{OR} \\ C \not\longrightarrow A \ \& \ C \longrightarrow B \end{array} \right\}$$

$\varepsilon > 0$

$$\phi^{\varepsilon}(A) = \min\left\{ |B| : \begin{array}{c} A \longrightarrow B \\ \mathrm{dist}(A,B) < \varepsilon \end{array} \right\}$$

$\boxed{\text{THM}}$ (N., POM)

FOR A CLASS $\mathcal{C}$

(1) $\mathcal{C}$ HAS ALL RESTRICTED DUALITIES

$\Updownarrow$

(2) $\sup_{A \in \mathcal{C}} \phi^{\varepsilon}(A) < \infty$

(FOR EVERY $\varepsilon > 0$)

dist DEFINES $\overline{\mathcal{H}}$ COMPLETION OF THE HOMOMORPHISM ORDER

DUALITIES IN $\overline{\mathcal{H}}$ CHARACTERIZED

$(\overline{F}, \overline{D})$ DUALITY IN $\overline{\mathcal{H}}$

$\Updownarrow$

EITHER $\overline{F}$ IS A CONNECTED GRAPH

OR $\overline{D}$ IS A MULTIPLICATIVE GRAPH

$\boxed{\text{REMARK}}$ DICHOTOMY BY COUNTING

$$\frac{\log \hom(F,G)}{\log |G|} \longrightarrow \{-\infty, 0, 1, \dots, \alpha(F), |F|\}$$

CLASS LIMIT SUPREMA
IN THE RESOLUTION

"THE LIMIT LOG-DENSITY" $< |F|$,

FOR EVERY NON-DISCRETE F

$$\Updownarrow$$

$\mathcal{C}$ NOWHERE DENSE

COMPARE CONVERGENCE BY
LOVASZ, SZEGEDY, SOS, ...

$\boxed{E}$ STRUCTURAL LIMITS

$\mathcal{L}$ FINITE RELATIONAL LANGUAG
(GRAPHS, DIGRAPHS, k-COLORED GRAPHS ...)

$FO(\mathcal{L})$ ALL FIRST-ORDER FORMULAS

$X \subseteq FO(\mathcal{L})$ A FRAGMENT

DEFINITION

$G_1, G_2, \dots, G_n, \dots$ IS **X-CONVERGENT**
IF FOR EVERY $\varphi \in X$ THE SEQUENCE
$\langle \varphi, G_1 \rangle, \langle \varphi, G_2 \rangle, \dots, \langle \varphi, G_n \rangle, \dots$ CONVERGES.

HERE:

$$\langle \varphi, G_n \rangle = \frac{\left| \{ (v_1, \dots, v_p) \mid G_n \models \varphi(v_1, \dots, v_p) \} \right|}{|G_n|^p}$$

$\varphi = \varphi(x_1, \dots, x_p)$ FORMULA WITH p FREE VARIABLES

$FO(\mathcal{L})$ BOOLEAN ALGEBRA

$\cup|$

X BOOLEAN SUBALGEBRA

$S(X)$ STONE SPACE OF X

(ULTRAFILTERS OR HOMOMORPHISMS $f : X \to \{0,1\}$)

\+ TOPOLOGY BY

$$K_X(\varphi) = \{ U \mid \varphi \in U \}$$
$$= \{ f \mid f(\varphi) = 1 \}$$

STONE DUALITY

$\boxed{\text{THM}}$

X SUBALGEBRA OF $FO(\mathcal{L})$.

FOR EVERY X-CONVERGENT SEQUENCE $(G_n)_{n \in \mathbb{N}}$ THERE EXISTS UNIQUE PROBABILITY MEASURE μ on $S(X)$ SUCH THAT FOR EVERY FORMULA $\varphi(x_1, \dots, x_p) \in X$ HOLDS:

$$\int_{S(X)} \mathbb{1}_{K(\varphi)}(x)\, d\mu(x) = \lim_{n \to \infty} \langle \varphi, G_n \rangle$$

SPECIAL FRAGMENTS

$QF(\mathcal{L})$	QUANTIFIER FREE
$FO_0(\mathcal{L})$	NO FREE VARIABLES (SENTENCES)
$FO_p(\mathcal{L})$	p - FREE VARIABLES
$FO^{local}(\mathcal{L})$	LOCAL FORMULAS

$FO_0(\mathcal{L})$ CONVERGENCE

$\Updownarrow$

ELEMENTARY CONVERGENCE

$(G_n)_{n\in\mathbb{N}}$ IFF FOR EVERY SENTENCE φ

ELEMENTAY CONV.

$$G_n \models \varphi$$

FOR ALL $n \geqslant n_0(\varphi)$.

QF - CONVERGENCE

$\Updownarrow$

L - CONVERGENCE

$(G_n)_{n \in \mathbb{N}}$ L - CONVERGENT

IFF

$\dfrac{\hom(F, G)}{|G|^{|F|}}$ CONVERGENT

FOR EVERY F

GRAPHS WITH $\Delta \leq D$

BS - CONVERGENCE (BENJAMINI SCHRAMM)

$\equiv$

FO_1^{local} - CONVERGENCE

($\equiv$ STATISTICS OF ISOMORPHISM TYPES OF NEIGHBORHOOD CONVERGES)

$\boxed{\text{THM}}$

$(G_n)_{n\in\mathbb{N}}$ WITH $\Delta(G_n) \leq D$

IS FO - CONVERGENT

$\Updownarrow$

(G_n) IS BS- CONVERGENT

$\not\Leftarrow$ FO_0- CONVERGENT

BACK TO

SPARSE VS DENSE

DICHOTOMY

LOVÁSZ: LIMITS FOR "INTERMEDIATE" CLASSES ?

FOR
CLASSES OF GRAPHS WITH
BOUNDED TREE DEPTH

ONE CAN DESCRIBE LIMIT OBJECTS.

HOPE FOR GENERAL BOUNDED
EXPANSION

VIA LOW-TREE-DEPTH
DECOMPOSITION.

CENTRAL PROBLEM OF
SPARSE GRAPHS

Jaroslav Nešetřil • Patrice Ossona de Mendez

Sparsity

Graphs, Structures, and Algorithms

Springer